# 2
# Stresses, strains, elasticity, and plasticity

## 2.1 Introduction

In many engineering problems we consider the behaviour of an *initially unstressed* body to which we apply some *first* load-increment. We attempt to predict the consequent distribution of stress and strain in key zones of the body. Very often we assume that the material is perfectly elastic, and because of the assumed linearity of the relation between stress-increment and strain-increment the application of a *second* load-increment can be considered as a separate problem. Hence, we solve problems by applying each load-increment to the unstressed body and superposing the solutions. Often, as engineers, we speak loosely of the relationship between stress-increment and strain-increment as a 'stress – strain' relationship, and when we come to study the behaviour of an inelastic material we may be handicapped by this imprecision. It becomes necessary in soil mechanics for us to consider the application of a stress-increment to a body that is *initially stressed,* and to consider the actual sequence of load-increments, dividing the loading sequence into a series of small but discrete steps. We shall be concerned with the changes of configuration of the body: each *strain-increment* will be dependent on the *stress* within the body at that particular stage of the loading sequence, and will also be dependent on the particular *stress-increment* then occurring.

In this chapter we assume that our readers have an engineer's working understanding of elastic stress analysis but we supplement this chapter with an appendix A (see page 293). We introduce briefly our notation for *stress* and *stress-increment,* but care will be needed in §2.4 when we consider *strain-increment.* We explain the concept of a *tensor* being divided into *spherical* and *deviatoric* parts, and show this in relation to the elastic constants: the axial compression or extension test gives engineers two elastic constants, which we relate to the more fundamental bulk and shear moduli. For elastic material the properties are independent of stress, but the first step in our understanding of inelastic material is to consider the representation of possible states of stress (other than the unstressed state) in *principal stress space.* We assume that our readers have an engineer's working understanding of the concept of 'yield functions', which are functions that define the combinations of stress at which the material yields plastically according to one or other theory of the strength of materials. Having sketched two yield functions in principal stress space we will consider an aspect of the theory of plasticity that is less familiar to engineers: the association of a plastic strain-increment with yield at a certain combination of stresses. Underlying this associated 'flow' rule is a *stability* criterion, which we will need to understand and use, particularly in chapter *5.*

## 2.2 Stress

We have defined the effective stress component normal to any plane of cleavage in a soil body in eq. (1.7). In this equation the pore-pressure $u_w$, measured above atmospheric pressure, is subtracted from the (total) normal component of stress $\sigma$ acting on the cleavage plane, but the tangential components of stress are unaltered. In Fig. 2.1 we see the total stress components familiar in engineering stress analysis, and in the following Fig. 2.2 we see the effective stress components written with *tensor-suffix* notation.

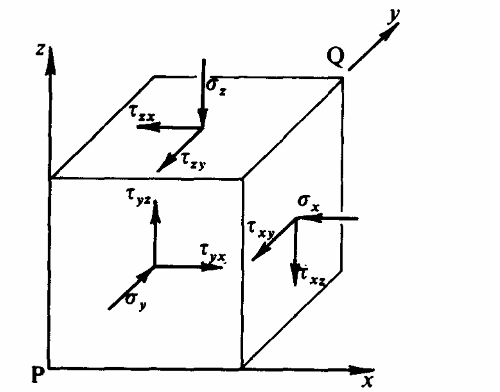


Fig. 2.1 Stresses on Small Cube: Engineering Notation

The equivalence between these notations is as follows:

$$\sigma_x = \sigma'_{11} + u_w \qquad \tau_{xy} = \sigma'_{12} \qquad \tau_{xz} = \sigma'_{13}$$
$$\tau_{yx} = \sigma'_{21} \qquad \sigma_y = \sigma'_{22} + u_w \qquad \tau_{yz} = \sigma'_{23}$$
$$\tau_{zx} = \sigma'_{31} \qquad \tau_{zy} = \sigma'_{32} \qquad \sigma_z = \sigma'_{33} + u_w.$$

We use *matrix* notation to present these equations in the form

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \begin{bmatrix} \sigma'_{11} & \sigma'_{12} & \sigma'_{13} \\ \sigma'_{21} & \sigma'_{22} & \sigma'_{23} \\ \sigma'_{31} & \sigma'_{32} & \sigma'_{33} \end{bmatrix} + \begin{bmatrix} u_w & 0 & 0 \\ 0 & u_w & 0 \\ 0 & 0 & u_w \end{bmatrix}.$$

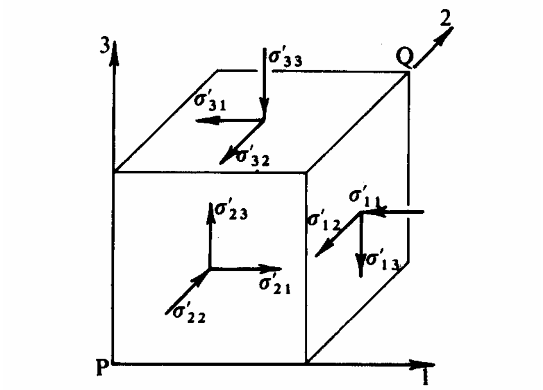


Fig. 2.2 Stresses on Small Cube: Tensor Suffix Notation

In both figures we have used the *same* arbitrarily chosen set of Cartesian reference axes, labelling the directions (x, y, *z*) and (1, 2, 3) respectively. The stress components acting on the cleavage planes perpendicular to the 1-direction are $\sigma'_{11}$, $\sigma'_{12}$ and $\sigma'_{13}$. We have exactly similar cases for the other two pairs of planes, so that each stress component can be written as $\sigma'_{ij}$ where the first suffix *i* refers to the direction of the normal to the cleavage plane in question, and the second suffix *j* refers to the direction of the stress component itself. It is assumed that the suffices *i* and *j* can be permuted through all the values 1, 2, and 3 so that we can write

$$\sigma'_{ij} = \begin{bmatrix} \sigma'_{11} & \sigma'_{12} & \sigma'_{13} \\ \sigma'_{21} & \sigma'_{22} & \sigma'_{23} \\ \sigma'_{31} & \sigma'_{32} & \sigma'_{33} \end{bmatrix}. \qquad (2.1)$$

The relationships $\sigma'_{ij} \equiv \sigma'_{ji}$ expressing the well-known requirement of equality of complementary shear stresses, mean that the array of nine stress components in eq. (2.1) is symmetrical, and necessarily degenerates into a set of only six independent components.

At this stage it is important to appreciate the sign convention that has been adopted here; namely, *compressive* stresses have been taken as *positive,* and the shear stresses acting on the faces containing the reference axes (through P) as *positive* in the *positive* directions of these axes (as indicated in Fig. 2.2). Consequently, the *positive* shear stresses on the faces through Q (i.e., further from the origin) are in the opposite direction.

Unfortunately, this sign convention is the exact opposite of that used in the standard literature on the Theory of Elasticity (for example, Timoshenko and Goodier[1], Crandall and Dahl[2]) and Plasticity (for example, Prager[3], Hill[4], Nadai[5]), so that care must be taken when reference and comparison are made with other texts. But because in soil mechanics we shall be almost exclusively concerned with compressive stresses which are universally assumed by all workers in the subject to be positive, we have felt obliged to adopt the same convention here.

It is always possible to find three mutually orthogonal *principal* cleavage planes through any point P which will have zero shear stress components. The directions of the normals to these planes are denoted by (*a, b, c*), see Fig. 2.3. The array of *three principal effective stress components* becomes

$$\begin{bmatrix} \sigma'_a & 0 & 0 \\ 0 & \sigma'_b & 0 \\ 0 & 0 & \sigma'_c \end{bmatrix}$$

and the directions (*a*, *b*, *c*) are called principal stress directions or principal stress axes. If, as is common practice, we adopt the principal stress axes as permanent reference axes we only require three data for a complete specification of the state of stress at P. However, we require three data for relating the principal stress axes to the original set of arbitrarily chosen reference axes (1, 2, 3). In total we require *six* data to specify stress relative to arbitrary reference axes.

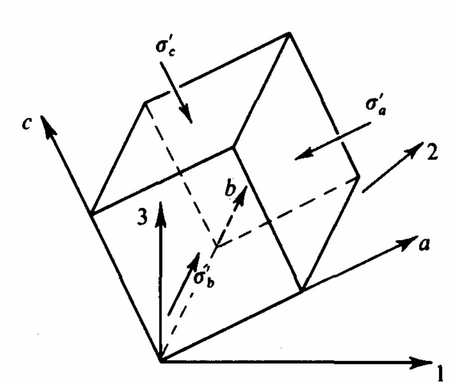

Fig. 2.3 Principal Stresses and Directions

## 2.3 Stress-increment

When considering the application of a small increment of stress we shall denote the resulting *change* in the value of any parameter $x$ by $\dot{x}$. This convention has been adopted in preference to the usual notation $\delta x$ because of the convenience of being able to express, if need be, a reduction in $x$ by $+\dot{x}$ and an increase by $-\dot{x}$ whereas the mathematical convention demands that $+\delta x$ always represents an increase in the value of $x$. With this notation care will be needed over signs in equations subject to integration; and it must be noted that a dot does *not* signify rate of change with respect to *time.*

Hence, we will write *stress-increment* as

$$\dot{\sigma}'_{ij} = \begin{bmatrix} \dot{\sigma}'_{11} & \dot{\sigma}'_{12} & \dot{\sigma}'_{13} \\ \dot{\sigma}'_{21} & \dot{\sigma}'_{22} & \dot{\sigma}'_{23} \\ \dot{\sigma}'_{31} & \dot{\sigma}'_{32} & \dot{\sigma}'_{33} \end{bmatrix}. \tag{2.2}$$

where each component $\dot{\sigma}'_{ij}$ is the difference detected in effective stress as a result of the small load-increment that was applied; this will depend on recording also the change in pore-pressure $\dot{u}_w$. This set of nine components of stress-increment has exactly the same properties as the set of stress components $\sigma'_{ij}$ from which it is derived. Complementary shear stress-increments will necessarily be equal $\dot{\sigma}'_{ij} \equiv \dot{\sigma}'_{ji}$ ; and it will be possible to *find* three principal directions *(d,e,f)* for which the shear stress-increments disappear $\dot{\sigma}'_{ij} \equiv 0$ and the three normal stress-increments $\dot{\sigma}'_{ij}$ become principal ones.

In general we would expect the data of principal stress-increments and their associated directions *(d,e,f)* at any interior point in our soil specimen to be six data quite independent of the original stress data: there is no a *priori* reason for their principal directions to be identical to those of the stresses, namely, *a,b,c*.

## 2.4 Strain-increment

In general at any interior point P in our specimen before application of the load-increment we could embed three extensible fibres PQ, PR, and PS in directions (1, 2, 3), see Fig. 2.4. For convenience these fibres are considered to be of unit length. After application of the load-increment the fibres would have been displaced to positions P'Q', P'R', and P'S'. This total displacement is made up of three parts which must be carefully distinguished:

(a) *body displacement*
*(b) body rotation*
*(c) body distortion.*

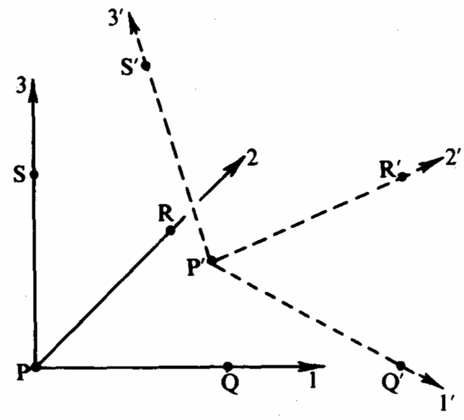


Fig. 2.4 Total Displacement of Embedded Fibres

We shall start by considering the much simpler case of two dimensional strain in Fig. 2.5. Initially we have in Fig. 2.5(a) two orthogonal fibres PQ and PR (of unit length) and their bisector PT (this bisector PT points in the spatial direction which at all times makes equal angles with PQ and PR; PT is not to be considered as an embedded fibre). After a small increment of plane strain the final positions of the fibres are P'Q' and P'R' (no longer orthogonal or of unit length) and their bisector P'T'. The two fibres have moved respectively through anticlockwise angles α and β, with their bisector having moved through the average of these two angles. This strain-increment can be split up into the three main components:

*(a) body displacement* represented by the vector PP' in Fig. 2.5(b);

*(b) body rotation of* $\dot{\theta}=\frac{1}{2}(\alpha+\beta)$ shown in Fig. 2.5(c);

*(c) body distortion* which is the combined result of *compressive strain-increments* $\dot{\varepsilon}_{11}$ and $\dot{\varepsilon}_{22}$ (being the shortening of the unit fibres), and a relative *turning* of the fibres of amount $\dot{\varepsilon}_{12}\equiv\dot{\varepsilon}_{21}\equiv\frac{1}{2}(\beta-\alpha)$, as seen in Fig. *2.5(d).*

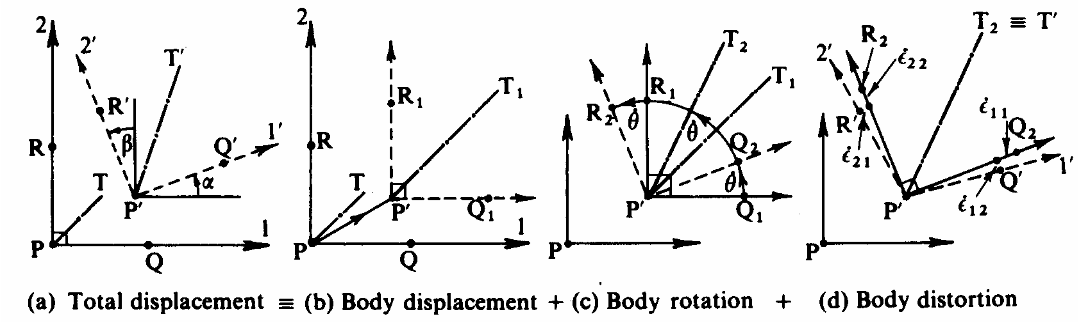


Fig. 2.5 Separation of Components of Displacement

The latter two quantities are the two (equal) *shear strain- increments* of irrotational deformation; and we see that their sum $\dot{\varepsilon}_{12}+\dot{\varepsilon}_{21}\equiv(\beta-\alpha)$ is a measure of the angular *increase* of the (original) right-angle between directions 1 and 2. The definition of shear

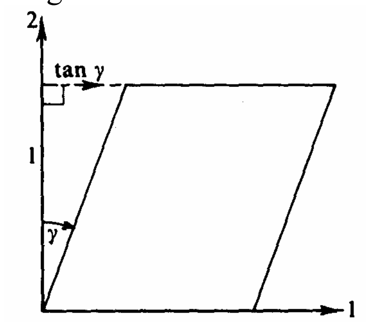


Fig. 2.6 Engineering Definition of Shear Strain

strain, γ,* often taught to engineers is shown in Fig. 2.6 in which $\alpha=0$ and $\beta=-\gamma$ and use of the opposite sign convention associates positive shear strain with a reduction of the right-angle. In particular we have $\dot{\theta}=-\frac{1}{2}\gamma=\dot{\varepsilon}_{12}=\dot{\varepsilon}_{21}$ and *half* of the distortion γ is really bodily rotation and only *half* is a measure of pure shear.

Returning to the three-dimensional case of Fig. 2.4 we can similarly isolate the body distortion of Fig. 2.7 by removing the effects of body displacement and rotation. The displacement is again represented by the vector PP' in Fig. 2.4, but the rotation is that experienced by the space diagonal. (The space diagonal is the locus of points equidistant from each of the fibres and takes the place of the bisector.) The resulting distortion of Fig. 2.7 consists of the compressive strain-increments $\dot{\varepsilon}_{11},\dot{\varepsilon}_{22},\dot{\varepsilon}_{33}$ and the associated shear strain-increments $\dot{\varepsilon}_{23}=\dot{\varepsilon}_{32},\dot{\varepsilon}_{31}=\dot{\varepsilon}_{13},\dot{\varepsilon}_{12}=\dot{\varepsilon}_{21}$ :and here again, the first suffix refers to the direction of the fibre and the second to the direction of change.

* Strictly we should use tan γ and not γ; but the definition of shear strain can only apply for angles so small that the difference is negligible.

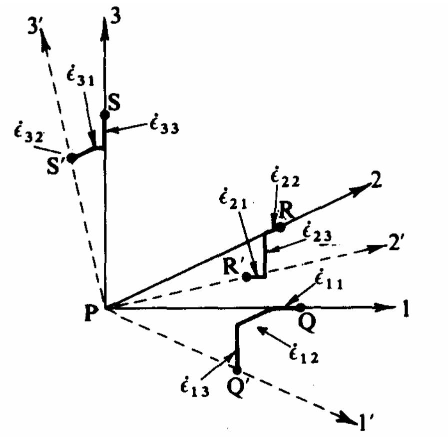


Fig. 2.7 Distortion of Embedded Fibres

We have, then, at this interior point P an array of nine strain measurements

$$\dot{\varepsilon}_{ij} = \begin{bmatrix} \dot{\varepsilon}_{11} & \dot{\varepsilon}_{12} & \dot{\varepsilon}_{13} \\ \dot{\varepsilon}_{21} & \dot{\varepsilon}_{22} & \dot{\varepsilon}_{23} \\ \dot{\varepsilon}_{31} & \dot{\varepsilon}_{32} & \dot{\varepsilon}_{33} \end{bmatrix} \tag{2.3}$$

of which only six are independent because of the equality of the complementary shear strain components. The fibres can be orientated to give directions ($g$, $h$, $i$) of *principal strain-increment* such that there are only compression components

$$\begin{bmatrix} \dot{\varepsilon}_g & 0 & 0 \\ 0 & \dot{\varepsilon}_h & 0 \\ 0 & 0 & \dot{\varepsilon}_i \end{bmatrix}.$$

The sum of these components $(\dot{\varepsilon}_g + \dot{\varepsilon}_h + \dot{\varepsilon}_i)$equals the increment of volumetric (compressive) strain $\dot{v}(= -\delta v)$which is later seen to be a parameter of considerable significance, as it is directly related to density.

There is no requirement for these principal strain-increment directions (*g, h, i*) to coincide with those of either stress (*a*, *b*, *c*) or stress-increment (*d, e, f*), although we may need to assume that this occurs in certain types of experiment.

## 2.5 Scalars, Vectors, and Tensors

In elementary physics we first encounter *scalar* quantities such as density and temperature, for which the measurement of a single number is sufficient to specify completely its magnitude at any point.

When *vector* quantities such as displacement $d_i$ are measured, we need to observe three numbers, each one specifying a component $(d_1, d_2, d_3)$ along a reference direction. Change of reference directions results in a change of the numbers used to specify the vector. We can derive a *scalar* quantity $d = \sqrt{\left(d_1^{\ 2} + d_2^{\ 2} + d_3^{\ 2}\right)} = \sqrt{\left(d_i d_i\right)}$ (employing the mathematical summation convention) which represents the distance or magnitude of the displacement vector *d,* but which takes *no* account of its direction.

Reference directions could have been chosen so that the vector components were simply (*d*, 0, 0), but then two direction cosines would have to be known in order to define the new reference axes along which the non-zero components lay, making three data in all. There is no way in which a Cartesian vector can be fully specified with less than three numbers.

The three quantities, stress, stress-increment, and strain-increment, previously discussed in this chapter are all physical quantities of a type called a *tensor.* In measurement of components of these quantities we took note of reference directions *twice,* permuting through them once when deciding on the cleavage planes or fibres, and a second time when defining the directions of the components themselves. The resulting arrays of nine components are symmetrical so that only six independent measurements are required. There is no way in which a symmetrical Cartesian tensor of the second order can be fully specified by less than six numbers.

Just as *one* scalar quantity can be derived from *vector* components so also it proves possible to derive from an array of *tensor* components *three* scalar quantities which can be of considerable significance. They will be independent of the choice of reference directions and unaffected by a change of reference axes, and are termed *invariants* of the tensor.

The simplest scalar quantity is the sum of the diagonal components (or trace), such as $\sigma'_{ii} = (\sigma'_{11}+\sigma'_{22}+\sigma'_{33}) = (\sigma'_a+\sigma'_b+\sigma'_c)$, derived from the stress tensor, and similar expressions from the other two tensors. It can be shown mathematically (see Prager and Hodge[6] for instance) that any strictly symmetrical function of all the components of a tensor must be an invariant; the first-order invariant of the principal stress tensor is $(\sigma'_a+\sigma'_b+\sigma'_c)$, and the second-order invariant can be chosen as $(\sigma'_b\,\sigma'_c+\sigma'_c\,\sigma'_a+\sigma'_a\,\sigma'_b)$ and the third-order one as $(\sigma'_a\,\sigma'_b\,\sigma'_c)$. Any other symmetrical function of a 3 × 3 tensor, such as $({\sigma'_a}^2+{\sigma'_b}^2+{\sigma'_c}^2)$ or $({\sigma'_a}^3+{\sigma'_b}^3+{\sigma'_c}^3)$, can be expressed in terms of these three invariants, so that such a tensor can only have three *independent* invariants.

We can tabulate our findings as follows:

| Array of | zero order | first order | second order |
|---|---|---|---|
| Type<br>Example | scalar<br>specific volume | vector<br>displacement | tensor<br>stress |
| Notation | υ | $d_i$ | $\sigma'_{ij}$ |
| Number of components | $3^0 = 1$ | $3^1 = 3$ | $3^2 = 9$ |
| Independent data | 1 | 3 | $\begin{cases} 9 \text{ in general} \\ 6 \text{ if symetrical} \end{cases}$ |
| Independent scalar quantities that can be derived | 1 | 1 | 3 |

## 2.6 Spherical and Deviatoric Tensors

**A** tensor which has only principal components, all equal, can be called *spherical.* For example, hydrostatic or spherical pressure $p$ can be written in tensor form as:

$$\begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix} \quad \text{or} \quad p\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad p\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}.$$

For economy we shall adopt the last of these notations. A tensor which has one principal component zero and the other two equal in magnitude but of opposite sign can be called *deviatoric.* For example, plane (two-dimensional) shear under complementary shear stresses $t$ is equivalent to a purely deviatoric stress tensor with components

$$t\begin{bmatrix}0 & & \\ & 1 & \\ & & -1\end{bmatrix}.$$

It is always possible to divide a Cartesian tensor, which has only principal components, into one spherical and up to three deviatoric tensors. The most general case can be divided as follows

$$\begin{bmatrix}\sigma'_a & & \\ & \sigma'_b & \\ & & \sigma'_c\end{bmatrix} = p\begin{bmatrix}1 & & \\ & 1 & \\ & & 1\end{bmatrix} + t_a\begin{bmatrix}0 & & \\ & 1 & \\ & & -1\end{bmatrix} + t_a\begin{bmatrix}-1 & & \\ & 0 & \\ & & 1\end{bmatrix} + t_c t_a\begin{bmatrix}1 & & \\ & -1 & \\ & & 0\end{bmatrix}$$

where

$$p = \frac{1}{3}\left(\sigma'_a + \sigma'_b + \sigma'_c\right), \qquad \left.\begin{aligned} t_a &= \frac{1}{3}\left(\sigma'_b - \sigma'_c\right) \\ t_b &= \frac{1}{3}\left(\sigma'_c - \sigma'_a\right) \\ t_c &= \frac{1}{3}\left(\sigma'_a - \sigma'_b\right) \end{aligned}\right\} \tag{2.4}$$

## 2.7 Two Elastic Constants for an Isotropic Continuum

A continuum is termed *linear* if successive effects when superposed leave no indication of their sequence; and termed *isotropic* if no directional quality can be detected in its properties.

The linear properties of an elastic isotropic continuum necessarily involve only *two* fundamental material constants because the total effect of a general tensor $\sigma'_{ij}$ will be identical to the combined effects of one spherical tensor *p* and up to three deviatoric tensors, $t_i$. One constant is related to the effect of the spherical tensor and the other to any and all deviatoric tensors.

For an elastic specimen the two fundamental elastic constants relating stress-increment with strain-increment tensors are (a) the *Bulk Modulus K* which associates a spherical pressure increment $\dot{p}$ with the corresponding specific volume change $\dot{v}$

$$\frac{\dot{p}}{K} = \left(\dot{\varepsilon}_a + \dot{\varepsilon}_b + \dot{\varepsilon}_c\right) = \frac{\dot{v}}{v} \tag{2.5}$$

and *(b)* the *Shear Modulus G* which associates each deviatoric stress-increment tensor with the corresponding deviatoric strain-increment tensor as follows

$$\begin{matrix}\text{stress}\\ \text{increment}\\ \text{tensor}\end{matrix} \quad \dot{t}\begin{bmatrix}0 & & \\ & 1 & \\ & & -1\end{bmatrix} \quad \begin{matrix}\text{gives rise}\\ \text{to strain}-\\ \text{increment}\\ \text{tensor}\end{matrix} \quad \frac{\dot{t}}{2G}\begin{bmatrix}0 & & \\ & 1 & \\ & & -1\end{bmatrix}. \tag{2.6}$$

(The factor of *2G* is a legacy from the use of the engineering definition of shear strain γ in the original definition of the shear modulus *t* = *Gγ*. We are also making the important assumption that the principal directions of the two sets of tensors coincide.)

It is usual for engineers to derive alternative elastic constants that are appropriate to a specimen in an axial compression (or extension) test, Fig. 2.8(a) in which $\dot{\sigma}'_a = \dot{\sigma}'_l\,; \dot{\sigma}'_b = \dot{\sigma}'_c = 0.$ *Young's Modulus E* and *Poisson's Ratio v* are obtained from

$$-\frac{\delta l}{l} = +\frac{\dot{l}}{l} = \dot{\varepsilon}_a = \frac{\dot{\sigma}'_l}{E} \quad \text{and} \qquad \dot{\varepsilon}_b = \dot{\varepsilon}_c = -\frac{\nu\dot{\sigma}'_l}{E}$$

which can be written as

$$\begin{bmatrix} \dot{\varepsilon}_a & & \\ & \dot{\varepsilon}_b & \\ & & \dot{\varepsilon}_c \end{bmatrix} = \frac{\dot{\sigma}'_l}{E}\begin{bmatrix} 1 & & \\ & -\nu & \\ & & -\nu \end{bmatrix}. \qquad (2.7)$$

By reference to Fig. 2.8(b) we can split this strain-increment tensor into its spherical and deviatoric parts as follows:

$$\begin{aligned}\begin{bmatrix} \dot{\varepsilon}_a & & \\ & \dot{\varepsilon}_b & \\ & & \dot{\varepsilon}_c \end{bmatrix} &= \frac{1}{3}(\dot{\varepsilon}_a + \dot{\varepsilon}_b + \dot{\varepsilon}_c)\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} + \frac{1}{3}(\dot{\varepsilon}_b - \dot{\varepsilon}_c)\begin{bmatrix} 0 & & \\ & 1 & \\ & & -1 \end{bmatrix} \\ &+ \frac{1}{3}(\dot{\varepsilon}_c - \dot{\varepsilon}_a)\begin{bmatrix} -1 & & \\ & 0 & \\ & & 1 \end{bmatrix} + \frac{1}{3}(\dot{\varepsilon}_a - \dot{\varepsilon}_b)\begin{bmatrix} 1 & & \\ & -1 & \\ & & 0 \end{bmatrix}.\end{aligned} \qquad (2.8)$$

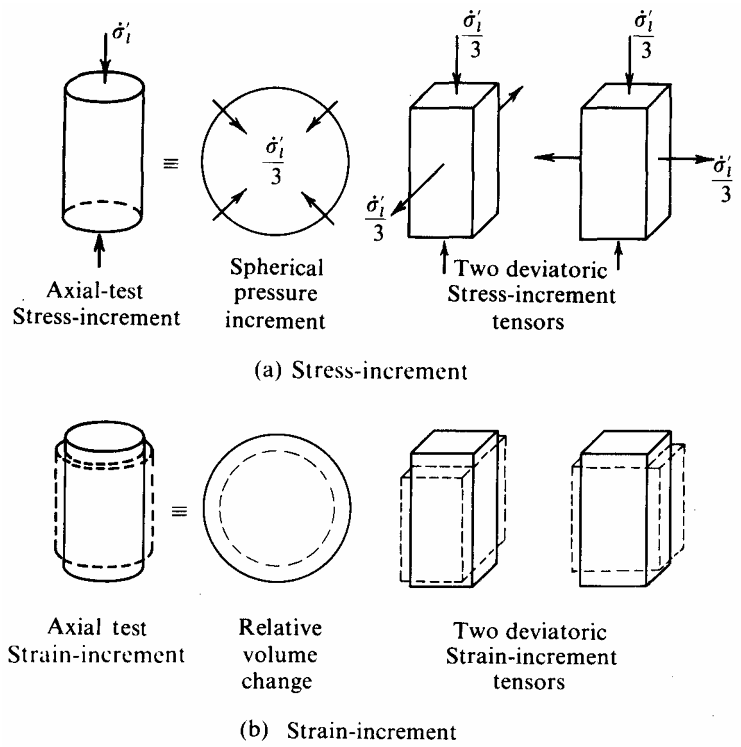


Fig. 2.8 Unconfined Axial Compression of Elastic Specimen

But from eq. (2.5)

$$\frac{\dot{\sigma}'_l}{3K} = \frac{\dot{\sigma}'_a + \dot{\sigma}'_b + \dot{\sigma}'_c}{3K} = \frac{\dot{p}}{K} = \dot{\varepsilon}_a + \dot{\varepsilon}_b + \dot{\varepsilon}_c$$

and from eq. (2.6)

$$\frac{1}{3}(\dot{\varepsilon}_b - \dot{\varepsilon}_c)\begin{bmatrix} 0 & & \\ & 1 & \\ & & -1 \end{bmatrix} = \frac{1}{6G}(\dot{\sigma}'_b - \dot{\sigma}'_c)\begin{bmatrix} 0 & & \\ & 1 & \\ & & -1 \end{bmatrix} + 2\,\text{similar expressions.}$$

Substituting in eq. (2.8) and using eq. (2.7) we have

$$\frac{\dot{\sigma}'_l}{E}\begin{bmatrix}1 & & \\ & -\nu & \\ & & -\nu\end{bmatrix}=\begin{bmatrix}\dot{\varepsilon}_a & & \\ & \dot{\varepsilon}_b & \\ & & \dot{\varepsilon}_c\end{bmatrix}=\frac{\dot{\sigma}'_l}{9K}\begin{bmatrix}1 & & \\ & 1 & \\ & & 1\end{bmatrix}-\frac{\dot{\sigma}'_l}{6G}\begin{bmatrix}-1 & & \\ & 0 & \\ & & 1\end{bmatrix}+\frac{\dot{\sigma}'_l}{6G}\begin{bmatrix}1 & & \\ & -1 & \\ & & 0\end{bmatrix}$$

which gives the usual relationships

$$\frac{1}{E}=\frac{1}{9K}+\frac{1}{3G} \text{ and } \frac{-\nu}{E}=\frac{1}{9K}-\frac{1}{6G} \tag{2.9}$$

between the various elastic constants.

We see that axial compression of *1/E* is only partly due to spherical compression *1/9K* and mostly caused by shearing distortion *1/3G*; conversely, indirect swelling *v/E* is the difference between shearing distortion *1/6G* and spherical compression *1/9K.* Consequently, we must realize that Young's Modulus alone cannot relate the component of a tensor of stress-increment that is directed across a cleavage plane with the component of the tensor of compressive strain-increment that gives the compression of a fibre embedded along the normal to that cleavage plane. An isotropic elastic body is *not* capable of reduction to a set of three orthogonal coil springs.

## 2.8 Principal Stress Space

The *principal* stresses $(\sigma'_a, \sigma'_b, \sigma'_c)$ experienced by a point in our soil continuum can be used as Cartesian coordinates to define a point D in a three-dimensional space, called principal stress space. This point D, in Fig. 2.9, although it represents the state of the particular point of the continuum which we are at present considering, only displays the *magnitudes* of the principal stresses and cannot fully represent the stress tensor because the three data establishing the *directions* of the principal stresses are not included.

The division of the principal stress tensor into spherical and deviatoric parts can readily be seen in Figs. 2.9 and 2.10. Suppose, as an example, the principal stresses in question are $\sigma'_a = 12, \sigma'_b = 6, \sigma'_c = 3;$ then, recalling eq. (2.4),

$$\begin{bmatrix}12 & & \\ & 6 & \\ & & 3\end{bmatrix}=\begin{bmatrix}\sigma'_a & & \\ & \sigma'_b & \\ & & \sigma'_c\end{bmatrix}$$

$$=\frac{(\sigma'_a+\sigma'_b+\sigma'_c)}{3}\begin{bmatrix}1 & & \\ & 1 & \\ & & 1\end{bmatrix}+\frac{(\sigma'_b-\sigma'_c)}{3}\begin{bmatrix}0 & & \\ & 1 & \\ & & -1\end{bmatrix}$$

$$+\frac{(\sigma'_c-\sigma'_a)}{3}\begin{bmatrix}-1 & & \\ & 0 & \\ & & 1\end{bmatrix}+\frac{(\sigma'_a-\sigma'_b)}{3}\begin{bmatrix}1 & & \\ & -1 & \\ & & 0\end{bmatrix}$$

$$=7\begin{bmatrix}1 & & \\ & 1 & \\ & & 1\end{bmatrix}+1\begin{bmatrix}0 & & \\ & 1 & \\ & & -1\end{bmatrix}+(-3)\begin{bmatrix}-1 & & \\ & 0 & \\ & & 1\end{bmatrix}+2\begin{bmatrix}1 & & \\ & -1 & \\ & & 0\end{bmatrix}$$

$$= \begin{bmatrix} 7 & & \\ & 7 & \\ & & 7 \end{bmatrix} + \begin{bmatrix} 0 & & \\ & 1 & \\ & & -1 \end{bmatrix} + \begin{bmatrix} 3 & & \\ & 0 & \\ & & -3 \end{bmatrix} + \begin{bmatrix} 2 & & \\ & -2 & \\ & & 0 \end{bmatrix}$$

$$= OA + AB + BC + CD = OD$$

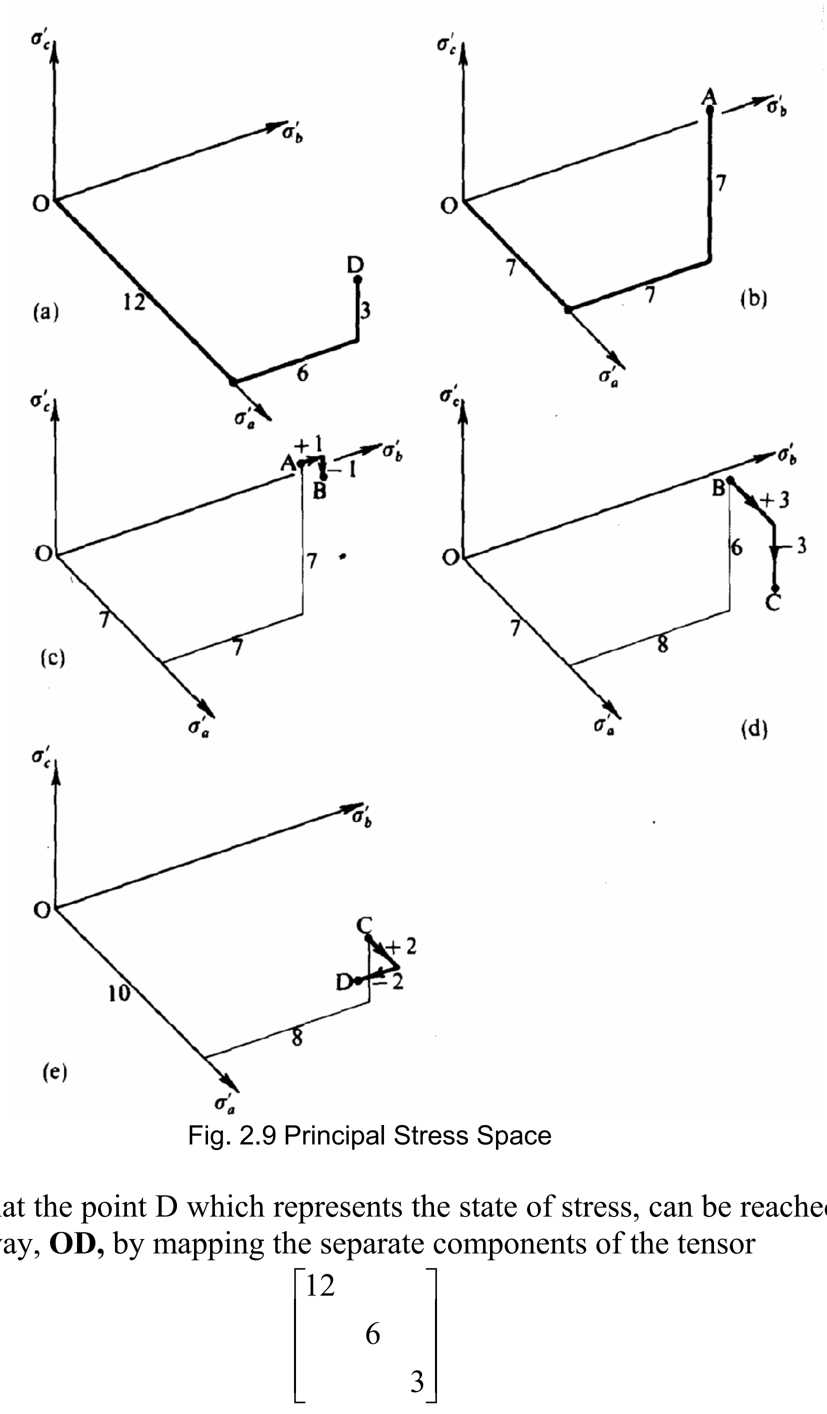


Fig. 2.9 Principal Stress Space

Hence, we see that the point D which represents the state of stress, can be reached *either* in a conventional way, **OD,** by mapping the separate components of the tensor

$$\begin{bmatrix} 12 & & \\ & 6 & \\ & & 3 \end{bmatrix}$$

or by splitting it up into the spherical pressure and mapping **OA**

$$\begin{bmatrix} 7 & & \\ & 7 & \\ & & 7 \end{bmatrix}$$

and three different deviatoric stress tensors and mapping **AB**, **BC,** and **CD:**

$$\begin{bmatrix} 0 & & \\ & 1 & \\ & & -1 \end{bmatrix}, \begin{bmatrix} 3 & & \\ & 0 & \\ & & -3 \end{bmatrix}, \text{and} \begin{bmatrix} 2 & & \\ & -2 & \\ & & 0 \end{bmatrix}.$$

So **AB** is a vector in the plane perpendicular to the u-axis and with equal and opposite components of unity parallel with the other two principal axes: and similarly **BC** and **CD** are vectors as shown.

As mentioned in §2.1, the principal stress space is particularly favoured for representation of theories of the yield strength of plastic materials. Experiments on metals show that large changes of spherical pressure p have no influence on the deviatoric stress combinations that can cause yield. Consequently, for perfectly plastic material it is usual to switch from the principal stress axes to a set of Cartesian axes *(x, y, z)* where

$$\left.\begin{aligned} x &= \frac{1}{\sqrt{3}}\left(\sigma'_a+\sigma'_b+\sigma'_c\right) = \sqrt{(3)}p \\ y &= \frac{1}{\sqrt{2}} = \left(\sigma'_b-\sigma'_a\right) \\ z &= \frac{1}{\sqrt{6}} = \left(2\sigma'_c-\sigma'_b-\sigma'_a\right). \end{aligned}\right\} \quad (2.10)$$

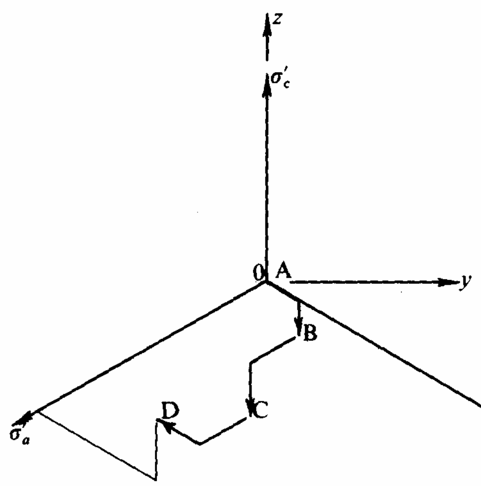


Fig. 2.10 Section of Stress Space Perpendicular to the Space Diagonal

The x-axis coincides with the *space diagonal;* change of spherical pressure has no influence on yielding and the significant stress combinations are shown in a plane *perpendicular* to the space diagonal. In the Fig. 2.10 we look down the space diagonal and see the plane *yz:* the *z*-axis is coplanar with the *x*-axis and the $\sigma'_c$-axis, but of course the three axes $\sigma'_a$, $\sigma'_b$ and $\sigma'_c$ are to be envisaged as rising out of the plane of Fig. 2.10. The mapping of the three different deviatoric stress tensors of our example is shown by the pairs of vectors **AB** and **BC** and **CD** in Fig. 2.10.

When we consider the yielding of perfectly plastic material the alternative theories of strength of materials can be either described by algebraic yield functions or described by symmetrical figures on this *yz*-plane, as we will now see in the next section.

## 2.9 Two Alternative Yield Functions

Two alternative yield functions are commonly used as criteria for interpretation of tests on plastic behaviour of metals. The first, named after Tresca, suggests that yield occurs when the maximum shear stress reaches a critical value *k*. We can see the effect of the criterion in the sector where in which the function becomes

$$F = \sigma'_a - \sigma'_c - 2k = 0 \tag{2.11}$$

and the intersection of this with the plane $p = \frac{1}{3}(\sigma'_a + \sigma'_b + \sigma'_c) = \text{const.}$ defines one side **IN** of the regular hexagon **INJLKM** in Fig. 2.11. The other sides are defined by appropriate permutation of parameters.

The second function, named after Mises, is expressed as

$$F = (\sigma'_b - \sigma'_c)^2 + (\sigma'_c - \sigma'_a)^2 + (\sigma'_a - \sigma'_b)^2 - 2Y^2 = 0 \tag{2.12}$$

where *Y* is the yield stress obtained in axial tension. This function together with $p = \frac{1}{3}(\sigma'_a + \sigma'_b + \sigma'_c) = \text{const.}$ has as its locus a circle of radius $\sqrt{\left(\frac{2}{3}\right)}Y$ in Fig. 2.11. Since these two loci are unaffected by the value of the spherical pressure $\frac{1}{3}(\sigma'_a + \sigma'_b + \sigma'_c) = p$, they will generate

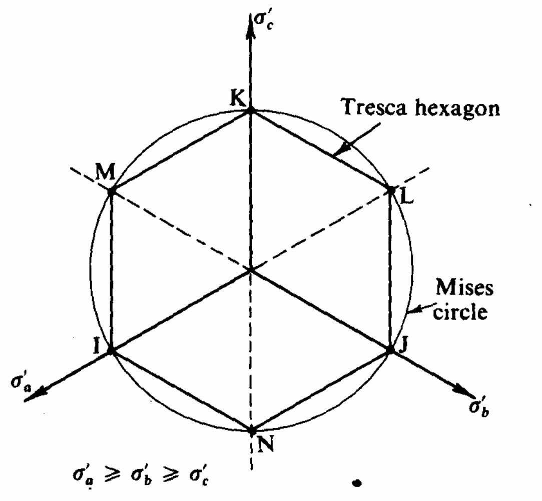


Fig. 2.11 Yield Loci of Tresca and Mises

for various values of *p* (or *x*) hexagonal and circular cylinders coaxial with the *x*-axis. These are illustrated in Fig. 2.12: these cylinders are examples of *yield surfaces,* and all states of stress at which one or other criterion allows material to be in stable equilibrium will be contained inside the appropriate surface.

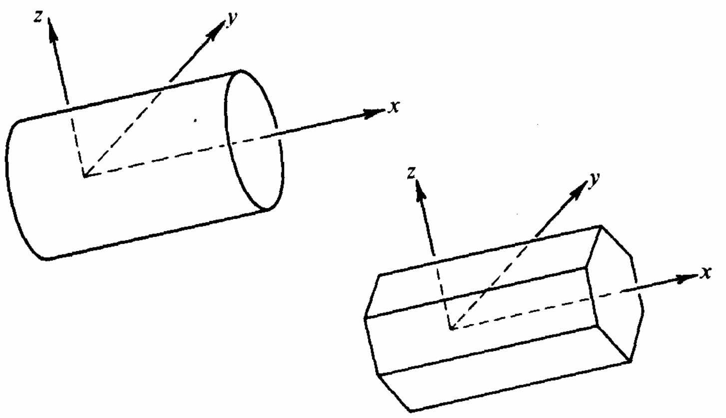


Fig. 2.12 Yield Surfaces in Stress Space

Most tests are what we will call *axial tests,* in which a bar or cylinder of material sustains two radial principal stresses of equal magnitude (often but not always zero) and the axial principal stress is varied until the material yields in compression or extension. Data of stresses in such axial tests will lie in an *axial-test plane* in principal stress space; the three diagonal lines **IL**, **JM**, **KN**, in Fig. 2.11, each lie in one of the three such planes that correspond to axial compression tests with $\sigma'_a$ *or* $\sigma'_b$ *or* $\sigma'_c$ respectively as the major principal stress. Now if we consider, for example, the axial-test plane for which $\sigma'_b = \sigma'_c$, this intersects Tresca's yield surface in the pair of lines $\sigma'_a - \sigma'_c = \pm 2k$, and it intersects Mises' yield surface in the pair of lines $\sigma'_a - \sigma'_c = \pm Y$. We cannot use axial-test data to decide which yield function is appropriate to a material – each will fit equally well if we choose 2k = *Y.* More refined tests* on thin-wall tubes of annealed metal in combined tension and torsion do appear to be fitted by Mises' yield function with rather more accuracy than by Tresca's yield function: however, the error in Tresca's function is not sufficient to invalidate its use in appropriate calculations.

## 2.10 The Plastic-Potential Function and the Normality Condition

As engineers, we concentrate attention on yield functions, because when we design a structure we calculate the factor by which all loads must be multiplied before the structure is brought to collapse. We use elastic theory to calculate deflections under working loads, and generally neglect the calculation of strain-increments in plastic flow. We will find in later chapters that our progress will depend on an understanding of plastic flow.

When any material flows without vorticity it is possible to find a *potential* function, such that the various partial derivatives of that function at any point are equal to the various velocity components at that point. We will meet 'equipotentials' when we discuss seepage in the next chapter, but the idea of a potential is not restricted to flow of water. It is the nature of plastic material to flow to wherever it is forced by the heavy stresses that bring the material to yield, so the potential function for plastic flow must be a function of the components r of stress. The classical formulation of theory of plasticity considers a class of materials for which the *yield* function *F(u)* also serves as the *plastic potential* for the flow. Each of the plastic strain-increments is found from the partial derivatives of the yield function by the equation (which is in effect a definition of plasticity)

$$\nu \dot{\varepsilon}^p_{ij} = \frac{\partial F}{\partial \sigma'_{ij}} \tag{2.13}$$

* An early set of tests was carried out in the engineering laboratories at Cambridge by G. I. Taylor and H. Quinney.[7]

where $v$ is a scalar factor proportional to the amount of work used in that particular set of plastic strain-increments.

If a material yields as required by Mises' function, eq. (2.12), we can calculate the gradients of this potential function as

$$v\dot{\varepsilon}_a = \frac{\partial F}{\partial \sigma'_a} = 6\left(\sigma'_a - \frac{\sigma'_a + \sigma'_b + \sigma'_c}{3}\right)$$

$$v\dot{\varepsilon}_b = \frac{\partial F}{\partial \sigma'_b} = 6\left(\sigma'_b - \frac{\sigma'_a + \sigma'_b + \sigma'_c}{3}\right)$$

$$v\dot{\varepsilon}_c = \frac{\partial F}{\partial \sigma'_c} = 6\left(\sigma'_c - \frac{\sigma'_a + \sigma'_b + \sigma'_c}{3}\right).$$

For given values of $(\sigma'_a, \sigma'_b, \sigma'_c)$ these equations fix the *relative* magnitudes of the strain-increments, but the number $v$ which adjusts their *absolute* magnitudes will depend on the amount of work used to force that particular set of plastic strain-increments. With given values of $(\sigma'_a, \sigma'_b, \sigma'_c)$ we can equally well associate a point in principal stress space on the yield surface: we can then visualize the plastic strain-increment vector as being normal to the yield surface at that point. Once we have decided upon a yield surface then the associated flow rule of the theory of plasticity obeys a *normality* condition: for Mises' yield function the plastic strain-increments are associated with vectors perpendicular to the cylindrical surface, while for Tresca's yield function the associated vectors are perpendicular to the faces of the hexagonal prism.

## 2.11 Isotropic Hardening and the Stability Criterion

In the yielding of a metal such as annealed copper we observe, as shown in Fig. 2.13, that once the material has carried an axial stress $Y$ it has hardened and will not yield again until that stress $Y$ is exceeded. We will be particularly interested in a class of *isotropic hardening* plastic materials, for which we can simply substitute the increasing values of $Y$ into equations such as (2.12) and get yield surfaces that expand symmetrically.

Our assumption of isotropic hardening does not mean that we dismiss an apparent occurrence of Bauschinger's effect. Some metal specimens, on hardening in axial tension to a stress $Y$, will yield on reversal of stress at an axial compressive stress less than $Y$, as illustrated in Fig. 2.13: in *metals* this indicates some anisotropy. In *soil* the yield strength is found to be a function of spherical pressure

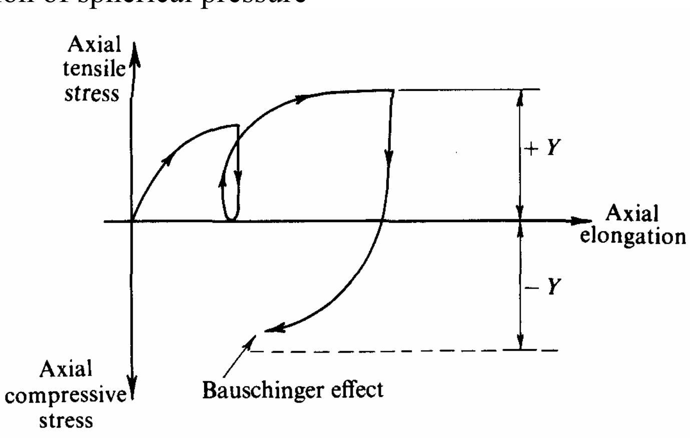


Fig. 2.13 Hardening in an Axial Test

and specific volume, so a major change of yield strength is to be expected on reversal of stress *without* anisotropy.

In Fig. 2.14(a) we have sketched a yield locus $F = 0$. The vector $\sigma'_{ij}$ represents a combination of stress that brings the material to the point of yielding. The fan of small vectors $\dot{\sigma}'_{ij}$ represent many possible combinations of stress-increment components which would each result in the same isotropic hardening of the material to a new yield locus $F' = 0$. In Fig. 2.14(b) we sketch a normal vector to the yield locus: no matter what stress-increment vector is applied the same associated plastic strain-increments will occur because they are governed by the particular stress combination that has brought the material to yield. The plastic strain-increments are not related directly to the stress-increments, nor are they directly proportional to the stress components (we can see in the figure that the strain- increment vector is *not* sticking out in the same direction as the extension of $\sigma'_{ij}$). The plastic strain-increments are found as the gradients of a potential function – the function is $F$ and $\dot{\varepsilon}^p_{ij}$ is normal to $F$.

Engineers have understandably been slow to accept that the materials with which they commonly work really do obey this curious associated flow rule. Recently, D. C. Drucker[8] has introduced the most persuasive concept of 'stability' which illuminates this matter. For all stress-increment vectors directed *outwards* from

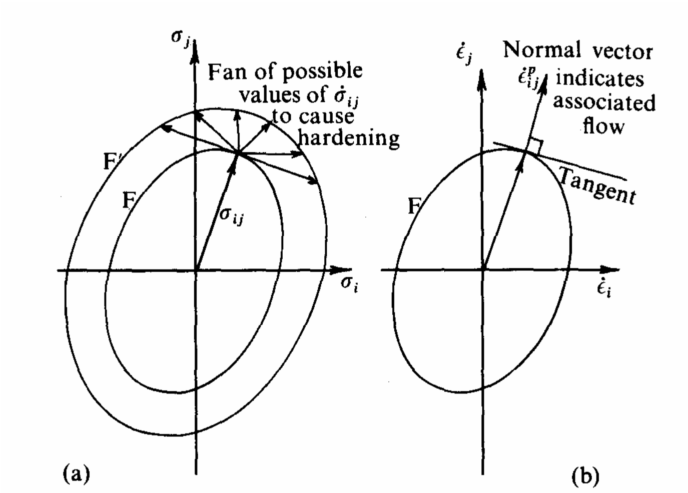


Fig. 2.14 Isotropic Hardening and Associated Plastic Flow

the tangent to the yield locus, the vector product of the stress-increment vector $\dot{\sigma}'_{ij}$ with the associated plastic strain-increment vector $\dot{\varepsilon}^p_{ij}$ will be positive or zero

$$\dot{\sigma}'_{ij}\,\dot{\varepsilon}^p_{ij} \geq 0. \qquad (2.14)$$

Plastic materials are *stable* in the sense that they only yield for stress increments that satisfy eq. (2.14). It is not appropriate for us now to make general statements that go further with the stability concept: it has been the subject of various discussions, and from here on we do best to develop specific arguments that are appropriate to our own topic. Our chapters *5* and 6 will pick up this theme again and go some way towards fulfilment of a suggestion of Drucker, Gibson, and Henkel[9], that soil behaviour can be described by a theory of plasticity.[10,11]

## 2.12 Summary

Most readers will have some knowledge of the theories of elasticity, plasticity and soil mechanics, so that parts of this chapter will already be familiar to them. As a consequence, the omission and the inclusion of certain material may seem curious on first reading, but the selection and emphasis are deliberate.

We are concerned with the development of a continuum analysis, so that we need to be clear about the manner in which stress and strain components can be defined in the interior of a granular body. It will be found that the current state of a soil depends on the stress and the specific-volume: stress is a second-order symmetrical tensor which requires six numbers for a definition, while specific volume is a scalar and is defined by one number. By emphasizing the importance of the elastic bulk modulus *K* and shear-modulus *G* we hope to develop a feeling for these tensor and scalar quantities.

For those who are familiar with Mohr's circle as a representation of stress it may be a surprise to find no mention of it in this chapter, although it will be required as an appendix to chapters 8 and 9. Its omission at this stage is deliberate on the grounds that Mohr's representation of stress imparts no understanding of the interrelation of stress-increment and strain-increment in elastic theory, that it plays little part in continuum theories, and that the uncritical use of Mohr's circle by workers in soil mechanics has been a major obstacle to the progress of our subject.

In contrast, the representation of stress $\sigma'_{ij}$, stress-increment $\dot{\sigma}'_{ij}$ and strain-increment $\dot{\varepsilon}_{ij}$, as compact symbols with the tensor suffix is helpful to our progress. Representation of stress in principal stress space is useful and gives an understanding of the difference between spherical pressure and the deviatoric stress tensors. A cautionary word is needed to remind our readers that when a point in principal stress space is defined by three numbers we necessarily assume that we know the three direction cosines of principal directions (needed to make up the six numbers that define a symmetrical 3×3 tensor). When we plot a stress-increment tensor in the same principal stress space, or associate a normal vector to a yield surface with the plastic strain-increment tensor, we necessarily assume that these tensors have the *same principal directions:* if not, then some more information is needed for the definition of these tensors. It will follow that the principal stress space representation is appropriate for discussion of behaviour of isotropic materials in which all principal directions coincide.

We have met yield surfaces that apply to the yielding of elastic/plastic metal. As long as the material is elastic the stress and strain are directly related, so the state of the metal at yield must be a function of stress, and the yield surface can be defined in principal stress space. When metals yield, only plastic distortion occurs, and there is no plastic volume change. The hardening of metal can be defined by a family of successive surfaces in principal stress space and the succession is a function of plastic distortion increment. However, soils and other granular materials show plastic volume change, and we will need to innovate in order to represent this major effect.

We have defined a parameter *p*, where

$$p = \frac{\sigma'_a + \sigma'_b + \sigma'_c}{3} \qquad \text{from eq. (2.4)}$$

gives an average or mean of the principal stress components, and *p* is called spherical pressure. One innovation that we will introduce is to propose that soil is a material for which the yield stress first increases and then decreases as spherical pressure increases.

*References to Chapter 2*

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